the deflection due to the moisture is constant over long periods of time, and hence the moisture does not affect the dynamic motion (Fig. 2b). The total response of the plate represented by Eq. (5) is shown in Fig. 2c. Varying the aspect ratio causes an increase in the magnitude of the plate center displacements and oscillations and a decrease in frequency (Fig. 3).

Concluding Remarks

From the results obtained for the particular plates considered, moisture will only affect the quasistatic response due to its very slow diffusion into the material. The dynamic response is a function of temperature only.

As the aspect ratio is varied from one, the center deflection increases dramatically, and the oscillation increases in magnitude while decreasing in frequency.

The results obtained showed good agreement with published data for both the graphite epoxy and aluminum plates.

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Analytical Approach to Free Vibration of Sandwich Plates

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Nomenclature

D = bending stiffness factor of the plate
E_f = Young's modulus of face sheets
G_c = shear modulus of the core
T_i = surface forces
u_i = displacements
X_i = volume forces
ε_{ij} = strains
λ^{*2} = nondimensional eigenvalue, $\omega a^2(\rho/D)^{1/2}$
ν = Poisson's ratio of the face material
ρ = mass per unit area
σ_{ij} = stresses
ω = circular frequency of vibration

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I. Introduction

SANDWICH plates usually consist of three layers of which two outer thin sheets are of high-strength material and a core is of low strength. In several applications, the thin skin sheets are made of aluminum, titanium, heat-resistant steel, or other materials such as plywood, hardboard, or reinforced plastics. The intermediate core serves to keep the facial layers apart and may also act as a thermal barrier.

Because of their high strength to weight ratios and good thermal and acoustical insulation properties, sandwich plate construction has become increasingly useful in various areas of structural design, especially in the aerospace industry, e.g., in fins, wings, and fuselage and pressure bulkheads.

In one of the earliest attempts at analysis, Reissner[1] developed the basic differential equations for the buckling of plates by using a simplified model consisting of two facings as membranes and a core resisting shear and normal stresses. Falgout[2] obtained the differential equations for free vibration of sandwich plates with isotropic faces and core by superposing the bending deflections and the deflections due to transverse shears. Several other authors used the finite element method to study the dynamic behavior of sandwich plates. Ahmed[3] employed the finite element technique to solve the vibration problem of a doubly curved honeycomb sandwich plate. Ng and Das[4] studied the vibration and buckling of clamped skew sandwich plates by the Galerkin method.

The purpose of this paper is to present an analytical approach, based on the reciprocal theorem,[5,6] for the free vibration of sandwich plates. It will be seen that the method works quite well, and excellent agreement is found between the results reported here and those published in earlier references.

II. Governing Equation

The theory presented here has been developed within the framework of linear theory and small displacements. Furthermore, we consider the face and core materials to be homogeneous and isotropic, the two face sheets to be of equal thickness t and the core layer to have a constant thickness c.

The differential equation for the free vibration of the plate in terms of the lateral displacement $w(x, y, t)$ can be written as[2]

$$\nabla^4 w(x, y, t) = \rho\left[\nabla^2 w(x, y, t)_{,tt}/(G_c C) w(x, y, t)_{,tt}/D\right] \quad (1)$$

where

$$\nabla^2 = (\cdot)_{xx} + (\cdot)_{,yy}, \qquad D = E_f t^3(1 + 3h^2/t^2)/[6(1 - v^2)]$$

$$h = c + t$$

If the plate is subjected to a time varying force $f(x, y, t)$, the equation governing the motion of the plate can be obtained by expressing the force and displacement as a product of two functions, one involving only spatial coordinates and the other involving the time t, as follows:

$$\nabla^4 W(x, y) + \lambda^4[(D/G^*)\nabla^2 W(x, y) - W(x, y)]$$
$$= F(x, y)/D - \nabla^2 F(x, y)/G^* \quad (2)$$

where $G^* = G_c c$ and frequency coefficient λ^2 can be written as $\lambda^2 = \omega\sqrt{(\rho/D)}$,

III. Reciprocal Theorem

The reciprocal theorem states that if a Hookean body is exposed to two different systems of volume and surface forces, then the actual work done by the forces of the first system along the displacements of the second system is equal to the work done by the forces of the second system along the displacements belonging to the first system.

Let there be two states of equilibrium, one given by $X_i^1, T_i^1, \sigma_{ij}^1, \varepsilon_{ij}^1$, and u_i^1, and the other by $X_i^2, T_i^2, \sigma_{ij}^2, \varepsilon_{ij}^2$, and u_i^2. The reciprocal theory just described is thus expressed as

$$\int_V \sigma_{ij}^1 \varepsilon_{ij}^2 \,\mathrm{d}V = \int_V \sigma_{ij}^2 \varepsilon_{ij}^1 \,\mathrm{d}V \tag{3}$$

Using the divergence theorem, we can rewrite Eq. (3) as

$$\int_S T_i^1 u_i^2 \,\mathrm{d}s + \int_V X_i^1 u_i^2 \,\mathrm{d}v = \int_S T_i^2 u_i^1 \mathrm{d}s + \int_V X_i^2 u_i^1 \mathrm{d}v \tag{4}$$

If a unit concentrated force applied at a point (ξ, η) acts on the first system and the volume forces of the first system are negligible, Eq. (4) may be written as

$$\Delta(\xi, \eta) + \int_S T_i^1 u_i^2 \mathrm{d}s = \int_S T_i^2 u_i^1 \mathrm{d}s + \int_V X_i^2 u_i^1 \mathrm{d}v \tag{5}$$

where $\Delta(\xi, \eta)$ is the displacement of the second system.

IV. General Solution

Consider a simply supported rectangular sandwich plate of dimensions $a \times b$. The plate is subjected to a concentrated harmonic force of unit intensity. The force has a circular frequency ω and is permitted to move on the surface of the plate so that its coordinates (ξ, η) are variable. In this case, Eq. (2) becomes

$$\nabla^4 W_1 + \lambda^4\left[(D/G^*)\nabla^2 W_1 - W_1\right] = \delta(x - \xi, y - \eta)/D \tag{6}$$

where $W_1(x, y, \xi, \eta)$ represents the displacement of the simply supported plate and $\delta(x-\xi, y-\eta)$ is the Dirac delta function satisfying Eq. (7)

$$\delta(x - \xi, y - \eta) = \begin{cases} \infty & \text{if } x = \xi, \quad y = \eta \\ 0 & \text{if } x \neq \xi, \quad y \neq \eta \end{cases} \tag{7}$$

A. Simply Supported Sandwich Plate

For this kind of boundary conditions, it is assumed that $W_1(x, y, \xi, \eta)$ has a Navier type solution,

$$W_1(x, y, \xi, \eta) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin K_m x \sin K_n y \tag{8}$$

where

$$K_m = m\pi/a, \qquad K_n = n\pi/b$$

The Dirac delta function $\delta(x-\xi, y-\eta)$ in Eq. (6) can be represented by a Fourier double sine series,

$$\delta(x - \xi, y - \eta) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{4}{ab} \sin K_m\xi \sin K_n\eta \sin K_m x \sin K_n y \tag{9}$$

After substituting Eqs. (8) and (9) into Eq. (6), we obtain

$$W_1(x, y, \xi, \eta) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{4}{abDK_{mn}} \sin K_m\xi \sin K_n\eta \sin K_m x \sin K_n y \tag{10}$$

where

$$K_{mn} = \left(K_m^2 + K_n^2\right)^2 - \lambda^4\left[D/G^*\left(K_m^2 + K_n^2\right) + 1\right]$$

B. Clamped Sandwich Plate

To generate the necessary moments along the boundaries to produce a fully clamped condition, the moments can be expressed in terms of series functions given by Eq. (11),

$$\begin{aligned} M_{x0} &= \sum_{n=1,2}^{\infty} A_n \sin K_n y, & M_{xa} &= \sum_{n=1,2}^{\infty} B_n \sin K_n y \\ M_{y0} &= \sum_{m=1,2}^{\infty} C_m \sin K_m x, & M_{yb} &= \sum_{m=1,2}^{\infty} D_m \sin K_m x \end{aligned} \tag{11}$$

Using the reciprocal theorem between the simply supported plate and the clamped plate, the displacement of the latter plate can be written as

$$W(\xi, \eta) = \int_0^b M_{x0}(W_{1,x})_{x=0} \,\mathrm{d}y + \int_0^b M_{xa}(W_{1,x})_{x=a} \,\mathrm{d}y + \int_0^a M_{y0}(W_{1,y})_{y=0} \,\mathrm{d}x + \int_0^a M_{yb}(W_{1,y})_{y=b} \,\mathrm{d}x \tag{12}$$

It is seen from Eq. (12) that the displacement $W(\xi, \eta)$ is expressed as a function of the coordinates (ξ, η). It may also be expressed as a function of coordinates (x, y) by replacing ξ and η in Eq. (12) with x and y.

Substituting Eqs. (10) and (11) into Eq. (12) and integrating, we obtain

$$W(\xi, \eta) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{2}{DK_{mn}}\left(\frac{K_m}{a}A_n + \frac{K_m}{a}B_n \cos m\pi + \frac{K_n}{b}C_m + \frac{K_n}{b}D_m \cos n\pi\right) \sin K_m\xi \sin K_n\eta \tag{13}$$

To determine unknown coefficients A_n, B_n, C_m, and D_m, Eq. (13) is subjected to the following boundary conditions valid for the clamped boundaries:

$$\begin{aligned} W_{,\xi}|_{\xi=0} &= 0, & W_{,\xi}|_{\xi=a} &= 0 \\ W_{,\eta}|_{\eta=0} &= 0, & W_{,\eta}|_{\eta=b} &= 0 \end{aligned} \tag{14}$$

Equations (14) indicate that the slopes taken normal to the four edges of the plate are zero. It is easy to prove that boundary conditions requiring displacements along the plate edges to be zero are satisfied automatically.

Substitution of Eq. (13) into Eq. (14) yields an eigenvalue system of homogeneous equations

$$[C]\{d\} = 0 \tag{15}$$

Here $[C]$ is a $4k \times 4k$ coefficient matrix, k is the number of terms used in the displacement solution, and $\{d\}$ is a column matrix containing A_n, B_n, C_m, and D_m. For a nontrivial solution of this set of $4k$ homogeneous algebraic equations, the determinant of the coefficient matrix of the unknown quantities A_n, B_n, C_m, and D_m must be zero. Since the eigenvalue λ^2 is involved in the coefficients of $A_n, \ldots, D_m$, the eigenvalues are obtained by searching for those values of λ^2 which cause the determinant of the coefficient matrix to vanish.

V. Computations and Results

The numerical results are obtained for rectangular plates having the following material properties: $E_f/G_c = 3.75$, $\upsilon = 0.3$, and $c/t = 10$. The aspect ratio ϕ is permitted to vary between 1 and 3. Two different values of the ratio a/c, i.e., 5 and 15.8, were selected to cover thin to thick plates. Computed eigenvalues are shown in Tables 1 and 2.

It should be pointed out that only 10 terms in Eqs. (15) are used to produce all of the results. Sample calculations were carried out with up to 20 terms in the equations, and it was found that there is

Table 1 Eigenvalues for clamped sandwich plate, $a/c = 5$ and $\lambda^{*2} = \omega a^2(\rho/D)^{1/2}$

	$\phi = b/a$								
Mode	1.0	1.25	1.5	1.75	2.0	2.25	2.5	2.75	3.0
1	32.32	27.33	24.94	23.66	22.92	22.46	22.15	21.94	21.78
2	58.51	44.25	36.31	31.31	28.68	27.77	25.48	24.57	23.92
3	78.51	55.36	53.38	44.35	38.36	34.26	31.37	29.29	27.75
4	90.58	67.56	62.11	58.84	50.99	44.39	39.59	36.02	33.32
5	125.34	80.34	73.64	60.25	52.46	52.12	49.60	44.42	40.41

Table 2 Eigenvalues for clamped sandwich plate, $a/c = 15.8$ and $\lambda^{*2} = \omega a^2(\rho/D)^{1/2}$

	$\phi = b/a$								
Mode	1.0	1.25	1.5	1.75	2.0	2.25	2.5	2.75	3.0
1	35.59	29.62	26.79	25.27	24.41	23.86	23.49	23.24	23.06
2	71.44	51.51	41.08	35.13	31.49	29.14	27.56	26.46	25.66
3	103.75	66.81	64.56	52.08	44.01	38.66	34.96	32.34	30.42
4	124.96	86.17	77.39	63.30	61.72	52.38	45.82	41.07	37.55
5	154.51	118.25	96.76	72.33	69.74	62.02	59.94	52.57	47.04

Table 3 Comparison of natural frequencies of clamped sandwich plate: $\omega = (\lambda^*/a)^2(D/\rho)^{1/2}$, $t = 0.4064$ mm, $c = 6.35$ mm, $a = 1016$ mm, $G_c = 6.895 \times 10^6$ Pa, $E_f = 6.896 \times 10^{10}$ Pa, $\rho = 5343.4$ Ns^2/m^4

	$\phi = 1.0$		$\phi = 1.5$		$\phi = 2.0$	
Mode	Present solution	Ref. 4	Present solution	Ref. 4	Present solution	Ref. 4
1	68.403	68.356	53.446	53.410	49.360	49.327
2	118.443	118.381	75.856	75.807	60.798	60.760
3	154.802	154.717	108.322	108.434	79.634	79.672

little change in the eigenvalues or mode shapes beyond 10 terms. It was, therefore, concluded that good convergence is obtained with 10 terms and all of the calculations presented in this paper are with $k = 10$.

To demonstrate the accuracy of the method comparisons between the data computed here and those given in Ref. 4 are made in Table 3. Results in Ref. 4 are based on Galerkin's method. It is found that there is good agreement between the computed results. Although only one example is shown here, it is obvious that other general boundary conditions can be handled the same way. They would lead to a new matrix $[C]$ in Eq. (15).

The approach is simple and quite efficient as the displacement solution is obtained in terms of definite integral, which are functions of boundary constraints and the corresponding displacement functions.

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New Evolutionary Direction Operator for Genetic Algorithms

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I. Introduction

GENETIC algorithms are search algorithms based on a natural evolution mechanism.[1] Recently, some aerodynamic optimization problems have been solved by genetic algorithms,[2–5] because genetic algorithms are robust and suitable for finding optimum effectively with a smaller possibility of falling in local optima than other algorithms. Gradient-based optimizers are another way to solve optimization problems. They are able to find a local optimum efficiently but are not superior to genetic algorithms from the viewpoint of global optimization.

To find global optima efficiently, the introduction of "evolutionary direction" into genetic algorithms is expected to be useful. A variety of methods to determine the evolutionary direction may be considered. One of them is a gradient-based optimizer.[6] However, gradient-based optimizers are usually time consuming because evaluation of gradients is necessary. In this paper, we propose a new operator for genetic algorithms to determine the evolutionary direction in a simple way. The operator does not require evaluation of gradients and thus is much less time consuming than the gradient-based optimizers.

II. Evolutionary Direction Operator

An optimization procedure by means of a genetic algorithm with an evolutionary direction is shown in Fig. 1. First, we prescribe the population size and select individuals of the population randomly (initialization). Then we evaluate the chromosomes and obtain fitness of each individual according to an evaluation function that is given a priori. In the case of optimization of aerodynamic configuration, for example, drag and/or lift evaluated by a computational

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